Distributed Command Governor Strategies for Constrained Coordination of Multi-Agent Networked Systems. Part I: Theory

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Abstract—In this paper a novel distributed FeedBack Command Governor (FB-CG) supervision strategy is presented for multi-agent networked systems subject to pointwise-in-time coordination constraints on the overall network evolutions. Specifically, a sequential distributed strategy where only one agent at a time is allowed to manipulate its own reference signal is fully detailed and its stability, feasibility and viability (liveness) properties rigorously proved. It is shown that, unlike the centralized case, liveness (the absence of deadlock situations) can be ensured only if the constraints satisfy a Constraints Qualification (CQ) condition. Several results and open problems in the liveness analysis are discussed and a counterexample is also provided for showing that for multi-input agents case the issue requires further investigations.

I. INTRODUCTION

The problem here addressed is the extension of the distributed supervision and coordination FeedForward Command Governor (FF-CG) strategies, determined in [1] and [2] for networked multi-agent systems in the presence of bounded persistent disturbances and subject to pointwise-in-time coordination constraints, to the more general class of FeedBack Command Governor (FB-CG) schemes.

FF-CG solutions [3] are mainly characterized by the fact that, unlike the standard FB-CG approach [4], their actions computation is not based on the current measure or estimate of the state and are generated by maintaining constant the FF-CG commands for several sampling steps, so as to enforce the system evolutions to stay close to space of steady-state feasible equilibria. Although this peculiarity of the FF-CG schemes make them an attractive solution for distributed frameworks because it alleviates the need to make the entire aggregate state, or substantial parts of it, known to all agents at each time instant, their tracking and coordination performance are sub-optimal when fast-varying reference signals are of interest and especially when bounded persistent disturbances are present.

A centralized solution to the supervision and coordination problem of interest here has been recently proposed in [5] in the quite general context depicted in Fig. 1. There, the master station is in charge of supervising and coordinating the slave systems via a data network. In particular, \( r_i \), \( y_i \), \( x_i \), \( u_i \) and \( c_i \) represent respectively: the nominal references, the feasible references, the states, performance-related and coordination-related outputs of the slave systems. In such a context, the supervision task can be expressed as the requirement of satisfying some tracking performance, viz. \( y_i \approx r_i \), whereas the coordination task consists of enforcing some pointwise-in-time constraints \( c_i \in C_i \) and/or \( f(c_1, c_2, \ldots, c_N) \in C \) on each slave system and/or on the overall network evolutions. To this end, the supervisor is in charge of modifying the nominal references into the feasible ones, when the tracking of the nominal references would produce constraint violations and hence loss of coordination. See [5] for motivations and examples of application.

In this paper we present a distributed supervision strategy for the above coordination problem in which the supervisory task is now distributed amongst several master agents which are assumed to be able to communicate amongst them and with the slave systems as well. Such a framework is depicted in Figure 2 and represents a particularly interesting solution in large-scale problems. In fact, it may be unrealistic to have a unique centralized coordination unit in many wide area networks while distributing the computation over a network of computing agents appears to be more natural. The proposed distributed approach differs from that presented in [1] and [2] because here the state is assumed to be available (with some time-delay due to the network latency) at the master agent sites and may be exploited for the distributed CG actions computation.

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A sequential distributed scheme is presented and its stability, feasibility and viability (liveness) properties fully investigated. It is important to remark that the introduction of the state in the play, unlike in [1] and [2], introduces many new technical challenges for the development of distributed schemes which cannot be overlooked and deserve carefully analysis. In this respect, this paper extends and makes clear several theoretical aspects of this sequential distributed scheme, not directly derivable from [1] and [2].

Novel insight to the liveness analysis discussed in [2] is provided here and remaining open issues discussed. In particular, a final counterexample is also presented that show that the notion of viability used, although instrumental for deriving simple numerical procedures for checking the liveness property and effective for single-input agents, is not enough in general in multi-input cases.

A complete case of study to a distributed voltage regulation in HV/MV smart-grids in the presence of Distributed Generation is presented in the companion paper [6].

II. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Let us consider a set of $N$ subsystems $A = \{1, \ldots, N\}$, each one being a LTI closed-loop dynamical system regulated by a local controller which ensures stability and good closed-loop performance in linear regimes when the constraints are not active (small-signal regimes when the coordination is effective). Let the i-th closed-loop subsystem be described by the following discrete-time model

$$
\begin{align*}
x_i(t+1) & = \Phi_i x_i(t) + G_i g_i(t) + G_{id}(t) + \sum_{j \in A^\neq (i)} \Phi_{ij} x_j(t) \\
y_i(t) & = H_i^y x_i(t) \\
c_i(t) & = H_i^c x_i(t) + L_i g_i(t) + L_{id}(t)
\end{align*}
$$

(1)

where: $t \in \mathbb{Z}_+$, $x_i \in \mathbb{R}^{n_i}$ is the state vector (which includes the controller states under dynamic regulation), $g_i(t) \in \mathbb{R}^{m_i}$, the CG action, which, if no constraints were present, would essentially coincide with the reference $r_i(t) \in \mathbb{R}^{m_i}$. The vector $d_i(t) \in \mathbb{R}^{n_{di}}$ is an exogenous bounded disturbance satisfying $d_i(t) \in \mathcal{D}_i$, $\forall t \in \mathbb{Z}_+$, with $\mathcal{D}_i$ a specified convex and compact set such that $0_{n_{di}} \in \mathcal{D}_i$; $y_i(t) \in \mathbb{R}^{n_{y}}$ the output, viz., a performance related signal. Moreover, the following matrices are defined. Finally, $c_i \in \mathbb{R}^{n_c}$ represents the local constrained vector which has to fulfill the set-membership constraint $c_i(t) \in C_i, \forall t \in \mathbb{Z}_+$, $C_i$ being a convex and compact polytopic set. It is worth pointing out that, in order to possibly characterize global (coupling) constraints amongst states of different subsystems, the vector $c_i$ in (1) is allowed to depend on the aggregate state and manipulable reference vectors $x = [x_1^T, \ldots, x_N^T]^T \in \mathbb{R}^n$, with $n = \sum_{i=1}^N n_i$, and $y = [y_1^T, \ldots, y_N^T]^T \in \mathbb{R}^m$, with $m = \sum_{i=1}^N m_i$, $d = [d_1^T, \ldots, d_N^T]^T \in \mathbb{R}^{n_d}$, with $n_d = \sum_{i=1}^N n_{di}$. Moreover, we denote by $r = [r_1^T, \ldots, r_N^T]^T \in \mathbb{R}^{n_r}$, $y = [y_1^T, \ldots, y_N^T]^T \in \mathbb{R}^{n_y}$ and $c = [c_1^T, \ldots, c_N^T]^T \in \mathbb{R}^{n_c}$, with $n_c = \sum_{i=1}^N n_c$, the other relevant aggregate vectors. The overall system arising by the composition of the above $N$ subsystems can be described as

$$
\begin{align*}
x(t+1) & = \Phi x(t) + G g(t) + G_{d} d(t) \\
y(t) & = H^y x(t) \\
c(t) & = H^c x(t) + L g(t) + L_{d} d(t)
\end{align*}
$$

(3)

where

$$
\begin{align*}
\Phi & = \begin{pmatrix} \Phi_{11} & \ldots & \Phi_{1N} \\ \vdots & \ddots & \vdots \\ \Phi_{N1} & \ldots & \Phi_{NN} \end{pmatrix}, G = \begin{pmatrix} G_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & G_N \end{pmatrix}, G_{d} = \begin{pmatrix} G_{d1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & G_{dN} \end{pmatrix}
\end{align*}
$$

$$
\begin{align*}
H^y & = \begin{pmatrix} H_{1y} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & H_{Ny} \end{pmatrix}, H^c = \begin{pmatrix} H_{1c} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & H_{Nc} \end{pmatrix}, L = \begin{pmatrix} L_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & L_N \end{pmatrix}, L_{d} = \begin{pmatrix} L_{d1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & L_{dN} \end{pmatrix}
\end{align*}
$$

It is further assumed that

A1. The overall system (3) is asymptotically stable.

A2. System (3) is off-set free i.e. $H^y(I_n - \Phi)^{-1}G = I_m$.

Roughly speaking, the CG design problem is that of locally determining, at each time instant $t$ and for each master agent $i \in A$ associated to each subsystem, a suitable reference signal $g_i(t)$ which is the best feasible approximation of $r_i(t)$ and such that its application never produces constraint violations, i.e. $c_i(t) \in C_i, \forall t \in \mathbb{Z}_+, i \in A$.

A centralized solution

A centralized solution of the above stated CG design problem have been proposed in [11]. There, at each time instant $t$, the overall CG action $g(t)$ is determined as a function of the current overall reference $r(t)$ and measured state $x(t)$

$$
g(t) := g(r(t), x(t))
$$

such that $g(t)$ is the best feasible approximation of $r(t)$ under $c(t) \in C$, where $C \subseteq \{C_1 \times \ldots \times C_N\}$ is the global admissible region. The solution is based on the following arguments: by linearity, one is allowed to separate the effects of the initial conditions and inputs from those of disturbances, e.g. $x(t) = \pi(t) + \tilde{x}(t)$, where $\pi(t)$ is the disturbance-free component and $\tilde{x}(t)$ depends only on the disturbances. Then, in the sequel we adopt the following notation

$$
\begin{align*}
\pi_y & := (I_n - \Phi)^{-1}G g \\
\pi_c & := H_y (I_n - \Phi)^{-1}G d + L g + L_{d} d
\end{align*}
$$

for the disturbance-free equilibrium solutions of (3) to a constant command $g(t) \equiv g$, with $g \in \mathbb{R}^{m}$.

Consider next the following set recursion

$$
\begin{align*}
C_0 & := \mathcal{C} \sim L_d D \\
C_k & := C_{k-1} \sim H_c \Phi^{k-1} G d D \\
& \vdots \\
C_\infty & := \bigcap_{k=0}^{\infty} C_k
\end{align*}
$$

(6)

where, for given sets $A, \mathcal{E} \subseteq \mathbb{R}^n, A \sim \mathcal{E} \subseteq A$ is a proper restriction of $A$ denoted in the literature as the $P$-difference between sets [10]:

$$
A \sim \mathcal{E} := \{x \in \mathbb{R}^n : x + e \in A, \forall e \in \mathcal{E} \}
$$

(7)

In [8] and [9] it has been proved that the sets $C_k$ are nonconservative restrictions of $C$ such that $\mathcal{F}(t) \in C_\infty, \forall t \in \mathbb{Z}_+$, implies that $c(t) \in C, \forall t \in \mathbb{Z}_+$. In the above references,
it has also been shown that the sets $C_k$ are the largest restrictions of $C$ such that $c(t) \in C_k$ implies $c(t + i) \in C$, for any $d(t + i) \in D$, $\forall i \in \{0, 1, \ldots, k - 1\}$. In particular, $c(t) \in C_{\infty}$, $\forall t \in \mathbb{Z}_+$, implies that $c(t) \in C$, $\forall t \in \mathbb{Z}_+$. Thus, one is allowed to consider the disturbance-free system evolutions only and adopt a "worst case" approach. For reasons which have been clarified in [4], it is convenient to introduce the following sets for a given $\delta > 0$

$$C_{\delta} := C_{\infty} \sim B_{\delta}$$

$$\mathcal{W}_{\delta} := \{ g \in \mathbb{R}^{2m} : \tau_g \in C_{\delta} \}$$

where $B_{\delta}$ is a ball of radius $\delta$ centered at the origin and $\mathcal{W}_{\delta}$ represents the set of all commands whose corresponding steady-state solutions $\tau_g$ satisfy the constraints with margin $\delta > 0$. The idea behind the CG approach is to select at each time instant $t$ an action $g(t)$ amongst all vectors of a suitable state depending subset of $\mathcal{W}_{\delta}$, each vector of which, if constantly applied as a command to the system from $t$ onwards, would give rise to system evolutions without constraint violations. Such a CG command is applied only on one sampling period after which a new state is measured and the procedure is repeated at the next time instant $t + 1$ on the basis of the new measurement $x(t + 1)$. Then, if we consider the following family of constant virtual command sequences

$$g(\cdot) = \{ g(k) : g(k) \equiv g \in \mathcal{W}_{\delta}, \; \forall k \in \mathbb{Z}_+ \}$$

the disturbance-free state $\tilde{x}(t)$ and $c$-vector evolutions emanating from $\tilde{x}(t)$ under a whatever constant command $g$ are given by

$$\begin{align*}
\tilde{x}(k, x(t), g) := & \Phi^k x(t) + \sum_{i=0}^{k-1} \Phi^{k-i-1} G g \\
\tilde{c}(k, x(t), g) := & H \tilde{x}(k, x(t), g) + L g
\end{align*}$$

where $\tilde{c}(k, x(t), g)$ has to be understood as the disturbance-free virtual evolution at the virtual time instant $k$ (opposite to the real time $t$) of the constrained vector $c$, from the initial condition $x(t)$ (applied at virtual time zero) under the constant command $g$. Finally, for any $k \geq 0$ we define the convex and closed set of admissible virtual sequences

$$\mathcal{V}(x, \tilde{k}) = \mathcal{W}_\delta \cap \mathcal{G}(x, \tilde{k})$$

where

$$\mathcal{G}(x, \tilde{k}) = \{ g \in \mathbb{R}^m : \tilde{c}(k, x, g) \in C_{\tilde{k}+\delta}, \forall k \in \mathbb{Z}_+ \}.$$  \hspace{0.5cm} (13)

Note that the case $\tilde{k} = 0$ is the standard one and describes the set of all constant virtual commands $g \in \mathcal{W}_\delta$ whose corresponding $c$-evolutions starting at time $k = 0$ from the current state $x$ satisfy the constraints $\forall k \in \mathbb{Z}_+$. As it will be clarified in the next section, the necessity of introducing $\tilde{k} > 0$ comes out for handling situations where time-delays and/or communication latency occur (see [7] for details).

Therefore, provided that $\mathcal{V}(\tilde{x}(t), 0)$ is nonempty, the centralized CG action can be chosen as the solution of the following constrained optimization problem

$$\tilde{g}(t) = \arg \min_{g \in \mathcal{V}(\tilde{x}(t), 0)} ||g - r(t)||_\Psi^2$$

where $\Psi_g = \Psi_g^T > 0_m$ and $\|v\|_\Psi := v^T \Psi v$. Such a solution represents the best feasible approximation of $r(t)$ which, if constantly applied from $t$ onwards never produces constraints violation.

### III. DISTRIBUTED SEQUENTIAL CG (S-CG)

Here we introduce a distributed CG scheme based on the above centralized CG approach. We assume the agents (the Master nodes in Fig. 2) connected via a communication network. Such a network is modeled with a communication graph: an undirected graph $G = (A, B)$, where $A$ denotes the set of the $N$ subsystems and $B \subset A \times A$ the set of edges representing the communication links amongst agents. More precisely, the edge $(i, j)$ belongs to $B$ if and only if the agents governing the $i$-th and the $j$-th subsystems are able to directly share information within one sampling time. The communication graph is assumed to be connected, i.e. for each couple of agents $i \in A, j \in A$ there exists at least one sequence of edges connecting $i$ and $j$, with the minimum number of edges connecting the two agents denoted by $d_{i,j}$. The set of all agents with a direct connection with the $i$-th agent will be referred to as the Neighborhood of the $i$-th agent $N_i = \{ j \in A : d_{i,j} = 1 \}$. We will assume that each agent acts as a gateway in redistributing data amongst the other, no directly connected, agents. Then, at each time instant $t$, each $i$-th agent is aware of the following vectors:

$$\xi_i(t - 1) := [g_i^T(t - d_{i,1}), \ldots, g_i^T(t - 1), g_i^T(t - d_{i,N})]^T \hspace{0.5cm} (12)$$

$$\phi_i(t - 1) := [x_i^T(t - d_{i,1}), \ldots, x_i^T(t), \ldots, x_i^T(t - d_{i,N})]^T$$

As a consequence, the most recent common information regarding the measurement of the overall state available to each agent is $x(t - d_{\max,i})$ where $d_{\max,i} = \max_{j \in A} d_{i,j}$.

Next, let us assume, without loss of generality, that the sequence $H = \{ 1, 2, \ldots, N - 1, N \}$ is a Hamiltonian cycle defined on $G$. The idea behind the approach is that only one agent per decision time is allowed to manipulate its local command signal $g_i(t)$ while all others are instructed to keep applying their commands. After a new CG computation, the agent in charge transmits its local command and state to the next updating agent that is necessarily a neighbor. Such a polling policy implies that, eventually after a preliminary initialization cycle, at each time instant $t$, the following information are available to the generic $i$-th "agent in charge"

- the history of the aggregate vectors applied in the last $N$ steps $g(t - N + j), j = 1, \ldots, N - 1$
- the measurement of the state at time $t - d_{\max,i}$, i.e. $x(t - d_{\max,i})$

By exploiting this information, it is possible for it to compute the estimation $\hat{x}(t)$ of the current free-disturbance state at time $t$ by means of the following recursions

$$\begin{align*}
\hat{x}(t - d_{\max,i}) = & \Phi \hat{x}(t - d_{\max,i}) \\
\hat{x}(k + 1) = & \Phi \hat{x}(k) + G g(k), k = t - d_{\max,i}, \ldots, t
\end{align*}$$

Then, by setting the parameter $\tilde{k}$ in (12) equal to the time delay $d_{\max,i}$, we can formulate the following distributed S-CG algorithm:
Sequencial-CG Algorithm (S-CG) - i-th Agent

AT EACH TIME $t$
1.1 IF $(t \ mod \ N) = = i$
1.1.1 RECEIVE $g(t-1)$ AND $x(t-1)$ FROM THE PREVIOUS AGENT IN THE CYCLE $\mathcal{H}$
1.1.2 SOLVE

$$g_i(t) = \arg \min_{\{w_i\}} \|w_i - r_i(t)\|_{\Psi_i}^2$$

SUBJECT TO :

$$[g_1^T(t-1), \ldots, g_N^T(t-1)]^T \in \mathcal{V}(\hat{e}(t), d_{\max})$$

(16)

1.1.3 APPLY $g_i(t)$
1.1.4 UPDATE $g(t) = [g_1^T(t-1), \ldots, g_i^T(t), \ldots, g_N^T(t-1)]^T$
1.2 ELSE
1.2.1 APPLY $g_i(t) = g_i(t-1)$
1.3 TRANSMIT ALL THE INFORMATION TO THE NEIGHBORS

where $\Psi_i > 0$ are weighting matrices, $t \ mod \ N$ is the remainder of the integer division $t/N$.

In order to present properties of the above algorithm let us introduce some definitions, notations and assumptions.

Definition 1 (Admissible direction) - Let a convex set $S \subset \mathbb{R}^m$ and a point $g \in S$. The vector $v \in \mathbb{R}^m$ represents an admissible direction for $g \in S$ if there exists a real $\lambda > 0$ such that $(g + \lambda v) \in S, \lambda \in [0, \lambda]$.

Definition 2 (Decision Set for the i-th agent) - The Decision Set $\mathcal{V}_i^S(g)$ for the i-th agent at a point $g \in S$ represents the set of all admissible directions belonging to $\mathbb{R}_i^m$ along which the agent could move in updating its action when all other agents hold their commands unvaried, viz. $\mathcal{V}_i^S(g) := \{d \in \mathbb{R}_i^m : [0_T, \ldots, 0_{i-1}, d_T, 0_{i+1}, \ldots, 0_N]^T \}$ is an admissible direction for $g \in S$.

Definition 3 (Viability property) - A point $g \in S$ is said to be "viable" if, for any admissible direction $v = [v_1^T, \ldots, v_N^T]^T \in \mathbb{R}^m$, $v_i \in \mathbb{R}^m_i$, with $\sum_{i=1}^N m_i = m$, at least one sub-vector $v_i \neq 0$ exists such that $v_i \in \mathcal{V}_i^S(g)$.

Definition 4 (Pareto Optimal Solution) - Let vectors $r_i, i = 1, 2, \ldots, N$ be given. Consider the following multi-objective problem:

$$\min_g ||g_1 - r_1||_{\Psi_1}^2, \ldots, ||g_i - r_i||_{\Psi_i}^2, \ldots, ||g_N - r_N||_{\Psi_N}^2$$

subject to $g = [g_1, \ldots, g_i, \ldots, g_N]^T \in \mathcal{W}_S$

(17)

A solution $g^* \in \mathcal{W}_S$ is a Pareto Optimal solution of the optimization problem (17) if there not exist $g \in \mathcal{W}_S$, such that: $||g_i - r_i||_{\Psi_i}^2 \leq ||g^*_i - r_i||_{\Psi_i}^2, \forall i \in \{1, \ldots, N\}$ and $||g_j - r_j||_{\Psi_j}^2 < ||g^*_j - r_j||_{\Psi_j}^2, j \in A$.

The above definitions are instrumental to characterize deadlock situations that, unlike the centralized solution, may exist in this decentralized scheme when the same constraint set $\mathcal{W}_S$ of the centralized scheme is used. The rationale is that by acting one agent at a time, certain viable paths existing in the centralized scheme are precluded and the agents could get stuck indefinitely without getting a Pareto Optimum. In order to clarify such matters, next Figg. 3-4 depict different viable and no-viable situations for points on the border of $\mathcal{W}_S$.

In order to avoid this deadlock situations we have to introduce the following assumption for the points belonging to the border of $\mathcal{W}_S$

A3. Each point belonging to $\partial(\mathcal{W}_S)$ is viable, $\partial(\mathcal{W}_S)$ denoting the border of $\mathcal{W}_S$.

In [2], the characterization of viable points, a computable way of checking if the viability property A3 is satisfied for a generic polyhedral set $\mathcal{W}_S$ and a geometrical method allowing one to compute suitable inner approximations of $\mathcal{W}_S$ satisfying A3 have been presented, along with a proof that all internal points of $\mathcal{W}_S$ are viable. Some new insight and open issues on the liveness analysis are given in the next section. Finally the following properties can be shown
to hold under A3 for the above stated S-CG scheme.

**Theorem 1:** Let assumptions A1-A3 be fulfilled. Consider system (3) along with the distributed S-CG selection rule (16) and let \( V(x(t), d_{\text{max}}) \) be non empty at time \( t = 0 \). Then:

1) for each agent \( i \in A \), at each decision time \( t \), the minimizer in (16) uniquely exists and can be obtained by locally solving a convex constrained optimization problem;
2) the overall system acted by the agents implementing the S-CG policy never violates the constraints, i.e. \( e(t) \in C \) for all \( t \in \mathbb{Z}_+ \);
3) whenever \( r(t) \equiv [r_1^T, \ldots, r_N^T]^T, \) with \( r_i \) a constant set-point, the sequence of solutions \( g(t) = [g_1^T(t), \ldots, g_N^T(t)]^T \) asymptotically converges to a Pareto-Optimal stationary (constant) solution of (17), which is given by \( r \) whenever \( r \in W_d \), or by any other Pareto-Optimal solution \( \hat{r} \in W_d \) otherwise.

**Proof**

1) The existence of an admissible solution for each agent at each decision time \( t \) can be proved by simply remarking that \( g_i(t) = g_i(t-1) \), is always an admissible, although not necessarily the optimal, solution for the prescribed problem at time \( t \).

2) At each time \( t \), from a centralized point of view, a command \( g(t) \) belonging to \( V(\hat{x}(t), d_{\text{max},t}) \), \( i \in A \) is applied to the overall plant. By construction, the latter set is equivalently characterizable as

\[
K(\hat{x}(t), d_{\text{max},t}) := \{ g \in W_d : e(k, \hat{x}(t), g) \} \bigoplus \bigcup_{j=1}^{d_{\text{max},t}} H_j \Phi^{-1} G_d D \oplus L_d D \subset C_k, \forall k \in \mathbb{Z}_+ \}
\]

where \( \bigoplus \) denotes the Pontryagin set sum defined as \( X + Y = \{ z = x + y \mid \forall x \in X, \forall y \in Y \} \). Moreover, at each time \( t \) the state \( \hat{x}(t) \), estimated by each acting \( i \)-th agent, satisfies

\[
\hat{x}(t) = x(t) - \sum_{j=1}^{d_{\text{max},t}} \Phi^{-1} G_d d(i) + L_d d(t)
\]

that ensures

\[
x(t) \in \hat{x}(t) \oplus \bigcup_{j=1}^{d_{\text{max},t}} \Phi^{-1} G_d D + L_d D.
\]

The latter implies that at each time instant a command \( g(t) \in V(\hat{x}(t), d_{\text{max},t}) \) is selected, and hence, by construction, this is sufficient to guarantee that \( e(t) \in C, \forall t \in \mathbb{Z}_+ \).

3) The stated convergence property follows simply because the sequences of solutions \( g_i(t) \) makes the sequences of local costs \( ||g_i(t) - r_i||^2_{\hat{q}_i} \) non increasing for any \( i = 1, \ldots, N \) under constant set-points. In fact, it is not convenient for the agents to modify their actual optimal solutions if the costs cannot be decreased further on. To this end, let \( g_i(t) \) be the S-CG action of the \( i \)-th agent at time \( t \), solution of the optimization problem (16). As already discussed, \( g_i(t) \) is still an admissible, though not necessarily the optimal, solution at time \( t + 1 \). Hence, the sequences of costs \( ||g_i(t) - r_i||^2_{\hat{q}_i} \) are all non-increasing, i.e.

\[
||g_i(t + 1) - r_i||^2_{\hat{q}_i} \leq ||g_i(t) - r_i||^2_{\hat{q}_i}
\]

Then, we want to show that any stationary optimal solution, viz. \( g(t) = g(t + 1) \) \( \forall t \), is Pareto Optimal by proving that a solution is not Pareto-Optimal if not stationary. To this end, let \( g^\prime(t) = [g_1^T(t), \ldots, g_N^T(t)]^T \) be the actual solution at time \( t \in \mathbb{Z}_+ \), which is assumed to be not Pareto-Optimal. The first thing to be noticed is that, similarly to what has been explicitly proved in [3], if the set-point \( g^\prime(t) \) in \( W_d \) were kept constant for a certain time interval \( \tau \), then the set \( G(x(t + \tau), \hat{k}) \) would finally contain a ball of finite radius \( \varepsilon > 0 \) centered in \( g^\prime(t) \). This means that viability losses cannot be never caused by \( G(x, \hat{k}) \) emptiness. Because of that and thanks to definition of \( V(x(t + \tau), \hat{k}) \) in (12), we may focus only on \( W_d \) in order to investigate on the viability properties of the proposed control scheme. Thus, being not \( g^\prime(t) \) Pareto Optimal, by definition, admissible directions exist which improve the costs. Supposedly, vectors \( v = [v_1^T, \ldots, v_N^T] \in \mathbb{R}^m \) would exist with \( g^\prime(t) + v \in W_d \), such that

\[
||g^\prime_i(t) + v_i - r_i||^2_{\hat{q}_i} - ||g^\prime_i(t) - r_i||^2_{\hat{q}_i} \leq 0,
\]

happens to hold for all \( i \in A^* := \{ i \in A : v_i \neq 0 \} \) with some of the above inequalities becoming strict for at least one index \( i \in A^* \). Because of the strict convexity of the norm \( || \cdot ||_{\hat{q}_i} \), the following inequality happens to be true for all \( \alpha \in (0, 1) \)

\[
||1 - \alpha||g^\prime_i(t) + \alpha (g^\prime_i(t) + v_i) - r_i||^2_{\hat{q}_i} - <(1 - \alpha)||g^\prime_i(t) - r_i||^2_{\hat{q}_i} + \alpha||g^\prime_i(t) + v_i - r_i||^2_{\hat{q}_i}
\]

Therefore, by means of straightforward algebraic manipulations, one arrives to

\[
||g^\prime_i(t) + \alpha v_i - r_i||^2_{\hat{q}_i} - ||g^\prime_i(t) - r_i||^2_{\hat{q}_i} < \alpha(||g^\prime_i(t) + v_i - r_i||^2_{\hat{q}_i} - ||g^\prime_i(t) - r_i||^2_{\hat{q}_i})
\]

for all \( \alpha \in (0, 1) \). Because (22), the right-hand term in (24) is always negative. Then, one can state

\[
||g^\prime_i(t) + \alpha v_i - r_i||^2_{\hat{q}_i} - ||g^\prime_i(t) - r_i||^2_{\hat{q}_i} < 0, \forall \alpha \in (0, 1)
\]

The latter may be interpreted as the fact that if the above admissible direction \( v \) did exist at \( g^\prime(t) \), for each agent \( i \in A^* \) it would be strictly convenient to move to \( g^\prime_i(t) + \alpha v_i \), for a suitable value of \( \alpha \), from its previous solution \( g^\prime_i(t) \). Now we have to verify that at least one agent is allowed to move from \( g^\prime_i(t) \) along \( v_i \) because of constraints. To this end, because of A3, note that \( v_i \) belongs to the \( i \)-th decision set \( \mathcal{V}_i(g^\prime(t)) \) for all agents corresponding to any non empty subset \( A_i^* \). Hence, according to the sequential S-CG updating policy, if at time \( t \), the index \( (t \ mod \ N) \in A_i^* \), then, because of (25), the agent \( i' = t \ mod \ N \) will find convenient to move into \( g^\prime_i(t) + \alpha v_i, \alpha \in [0, \alpha] \). In fact, because of viability of \( g^\prime(t) \) (see the above definition)
Definition 5 (Cost-Descentent Direction) Let $S \subset R^n$ be a convex set and consider a point $g \in S$. The admissible direction $v \in R^n$ at $g$ represents a Cost-Descentent direction for a given reference $r = [r_1^T, ..., r_N^T]^T$ with $\sum_{i=1}^N m_i = m$, if $\lambda \geq g_i + v_i \lambda - r_i$ $\parallel \phi_j \parallel_\phi \forall i \in \{1, ..., N\}, 0 < \lambda \leq \lambda$ and $\parallel g_j - r_j \parallel_\phi > \parallel g_j - v_j \lambda - r_j \parallel_\phi \forall j \in A, 0 < \lambda \leq \lambda$. □

Definition 6 (Reference Dependent Viability) A point $g \in S$ is said to be “viable” if, for all desired references $r = [r_1^T, ..., r_N^T]^T \in R^n$, $r_i \in R^{m_i}$ with $\sum_{i=1}^N m_i = m$ and $r \neq g$, at least one subvector $v_i \neq 0, i \in A$ exists such that $v_i \in V_i(g)$ and the direction $[0_{i1}^T, ..., v_i, ..., 0_{iN}^T]$ is a Cost-Descentent direction at $g$ w.r.t $r$. □

One of the merits of viability notion given by Definition (3) is that it gives rise to simple numerical procedures to test Assumption A3. On the contrary, the notion of viability based on Definition (6) is expected to be more cumbersome to be verified and the issue requires further investigations. The numerical test presented in [2], here recalled for the sake of the understanding, checks a finite number of points belonging to $\partial (V(g))$ in order to verify Assumption A3. Such a test is necessary and sufficient to state that: if a point of $\partial (V(g))$ is viable according to Definition (3) because the sub-vectors $v_1$ and $v_2$ of the admissible direction $v$ do not belong to $V(g)$, then, Assumption A3 is not satisfied. Nevertheless, the polyhedron $S$ is viable in $g'$ and no deadlocks occur. In fact, although agent 2 is locked, agent 1 can move along $v_1'$ for improving its cost.

Fig. 5. Sufficiency of Assumption A3. a) 3D view of the investigated polyhedron $S$. b) $S$ projection on the $g_1 - g_2$ plane. In this case, there are two agents $A = \{1, 2\}$ minimizing respectively $\parallel g_1 - v_1 \parallel_\phi + \parallel g_2 - r_2 \parallel_\phi$ and $\parallel g_1 - r_1 \parallel_\phi$. The point $g'$ is not viable according to Definition (3) because $v_1'$ and $v_2$ of the admissible direction $v$ do not belong to $V(g)$ and, respectively, $V_1(g')$. Then, Assumption A3 is not satisfied. Nevertheless, the polyhedron $S$ is viable in $g'$ and no deadlocks occur.
respectively, vector $b$.

A further very challenging issue already discussed in [2] is represented by the achievement of arbitrarily accurate multi-box inner approximations of the original constraint set which satisfy the viability property. A method was presented in [2], directly adaptable to the present context, which is proved to satisfy Assumption A3 in the case that all agents have mono-dimensional decision sets, viz. $m_i = 1, \forall i \in \mathcal{A}$ and $N = m$.

Numerical simulations suggest that this method makes the control strategy to converge to a Pareto-optimum also in the case of multi-dimensional problems $m_i \geq 1, \forall i \in \mathcal{A}$. The investigation of this conjecture is a current research argument whose partial results are hereafter discussed. In particular, we will detail that it is possible to prove that such a claim cannot be given in terms of the viability Definition (3) because it is possible to build a counterexample (Example 1 of the next subsection). This fact encourages the investigation of less restrictive notions of viability, such as the one given in Definition (6).

In order to fully understand the counterexample, the key ideas of the multi-box approximation method are here briefly recalled. Such an approximation is obtained by partitioning the original non viable set $\mathcal{S}$ by means of an arbitrary number of boxes obtaining a multi-box inner approximation $\mathcal{M}(\mathcal{S}) \subset \mathcal{S}$, as depicted in Fig. IV for a specific two-dimensional case. Then, the inner-approximating set is computed as the convex hull of $\mathcal{M}(\mathcal{S})$, say it $\mathcal{S}' := \text{co} [\mathcal{M}(\mathcal{S})]$. Such a set is viable by construction and enjoys the following properties:

**Proposition 1:** Let $\mathcal{S} \subset \mathbb{R}^m$ be expressed as the intersection of $|\mathcal{J}|$ inequalities $\mathcal{A} \mathcal{G} \leq b$. Then, for each point $y$ of the border of $\mathcal{S}'$, say it $\partial(\mathcal{S}')$, the next two properties hold true

1) at $y$ there exist $m$ admissible directions aligned to the axes, i.e. $g + [0, \ldots, \lambda \mathcal{J}_1, \ldots, 0] T \in \mathcal{S}'$, $\lambda \in [0, \tilde{\lambda}], \forall i \in \{-1,1\}, \forall i \in \mathcal{A}, \forall g \in \partial(\mathcal{S}')$.

2) for all $s \in \mathbb{R}^m$ such that $\mathcal{A}(g + s) \leq b$, the following condition is satisfied

$$A(g + [0_1, \ldots, s_i, \ldots, 0_m] T) \leq b$$

for at least an index $i \in \mathcal{A}$.  


In [2] it has been proved that the viability of $\mathcal{S}'$ is ensured by the second item of Proposition 1.

Then, in our context, a no viable polyhedron $\mathcal{W}_g$ can be always approximated with a viable multi-box polyhedron $\mathcal{W}'_g$ and the S-CG problem recast as follows

$$g_i(t) = \arg \min_{g_i} \| g_i - r_i(t) \|_2^T$$

subject to: 

$$\{ g(t) = [g_1^T(t - \tau), \ldots, g_i^T(t - \tau), \ldots, g_N^T(t - \tau)] T \in \mathcal{V}'(\hat{x}(t), d_{\text{max}}) \}$$

where

$$\mathcal{V}'(x, \hat{k}) := \mathcal{W}'_g \cap \mathcal{G}(x, \hat{k}).$$

**Goal of this subsection is to show two interesting examples of use of the above presented multi-box inner approximation technique in the case of systems with multi-dimensional decision sets. The first is the above mentioned counterexample showing the fact that the convex hull of the multi-box approximation set may fail to be viable in general multi-dimensional problems, when the viability notion of Definition 3 is adopted. The second example is instead provided in order to remark the fact that this method may however be used as a "first attempt" to obtain an approximated viable set of a constrained region, because in many cases it is a successful approach.**

**1) Example 1:** Consider a two-agent case each one acting on two-dimensional local space ($m_1 = m_2 = 2$) with $m = 4$. We are looking for two hyper-planes

$$\begin{cases}
a_1^T g = b_1 \\
a_2^T g = b_2
\end{cases}$$

for which the Test of Lemma 1 successes. Because we assume that such hyper-planes arise from the multi-box approximating procedure, we consider that they represent faces of the convex hull of such an approximation which enjoy the properties stated in Proposition 1. In order to simplify our search, without any loss of generality, we look for a direction $w$ having always the same coordinate in all axes, i.e. $w = [v, v, v, 0] T$ where $v \in \mathbb{R}$. The success of the Test in Lemma 1 suggests us that $w$ and the coefficient vectors $a_1$ and $a_2$ have to jointly satisfy the following conditions

$$\begin{cases}
(a_1 + a_2 + a_1^2 + a_2^2) v < 0 \\
(a_1 + a_2 + a_1^2 + a_2^2) v < 0 \\
(a_1 + a_2) v > 0 \\
(a_1 + a_2) v < 0 \\
(a_2 + a_2) v < 0 \\
a_1 + a_2 > 0
\end{cases}$$

In fact, according to Lemma 1, the latter indicates that any $g$ lying on the considered hyperplane is not viable. Moreover, we have to add further constraints in order to impose that such hyperplane belongs to the convex hull of a multi-inner box approximation and then satisfies Proposition 1. In particular, we impose that the first component of $w$ satisfies
point (2) of Proposition 1. Furthermore, we have also to consider the following additional conditions
\[
\begin{align*}
1_a^1v < 0 \\
1_a^2v < 0
\end{align*}
\]  
(31)

Because the other components of \( w \) satisfy point (1) of Proposition 1, the following constraints have to be included in our search
\[
\begin{align*}
|a_1^2v < 0 \land a_2^2v < 0 \lor (-a_1^2v < 0 \land -a_2^2v < 0) \\
|a_1^3v < 0 \land a_2^3v < 0 \lor (-a_1^3v < 0 \land -a_2^3v < 0) \\
|a_1^4v < 0 \land a_2^4v < 0 \lor (-a_1^4v < 0 \land -a_2^4v < 0)
\end{align*}
\]  
(32)

Observe that several solutions for the system of inequalities (30)-(32) exist and may be represented by the following compact form
\[
\begin{align*}
1_a^2 < 0 \\
1_a^3 > -a_2^2 \\
|a_1^3 < -a_1^2 - a_2^2 \land 0 < a_4^2 < -a_1^3 - a_2^3 \land a_2^3 < 0 \lor \\
0 < a_2^3 < -a_1^2 - a_2^2 \land 0 < -a_2^3 < a_1^2 - a_2^3 \\
- a_2^3 \lor v > 0 \lor a_2^1 > 0 \lor \\
|a_1^2 < -a_1^2 - a_2^2 \land a_2^1 < 0 \lor a_1^2 - a_2^2 \land 0 < a^2 \lor - a_1^2 \land \\
(0 < a_2^1 < -a_2^2) \land (-a_2^1 < a_2^2 < 0 \land v > 0)
\end{align*}
\]  
(33)

where \( \land \) and \( \lor \) represents logic AND and, respectively, OR conditions. As a conclusion, from the success of the Test of Lemma 1 is not possible to guarantee that the multi-box approximation is viable in this simple multi-dimensional case.

2) Example 2: As a second scenario, we consider the same problem of Example 1 with one of the two agents acting on a scalar decision space, i.e. \( m_1 = 1 \) and \( m_2 = 3 \). In this case we look for two hyper-planes and an admissible direction \( w = [w_1, w_2] \), \( w_1 \in \mathbb{R} \), \( w_2 \in \mathbb{R}^2 \) that satisfy the Test of Lemma 1 or, equivalently, the following system of inequalities
\[
\begin{align*}
1_a^1w < 0 \\
1_a^2w < 0 \\
1_a^3w_1^2 + a_1^3w_2^2 + a_1^3w_3^2 < 0 \\
1_a^2w_1^2 + a_1^2w_2^2 + a_1^2w_3^2 > 0 \\
1_a^4w_1 > 0 \\
1_a^4w_2 < 0
\end{align*}
\]  
(34)

Furthermore, an additional constraint of type (31) has to be added for the satisfaction of property (2) of Proposition 1, that is
\[
\begin{align*}
1_a^1w_2 < 0 \\
1_a^2w_2 < 0
\end{align*}
\]  
(35)

and further constraints of type (32) are needed in order to take into account point (1) of Proposition 1
\[
\begin{align*}
|a_1^w_1 < 0 \land a_2^w_1 < 0 \lor (-a_1^w_1 < 0 \land -a_2^w_1 < 0) \\
|a_1^w_2 < 0 \land a_2^w_2 < 0 \lor (-a_1^w_2 < 0 \land -a_2^w_2 < 0) \\
|a_1^w_2 < 0 \land a_2^w_2 < 0 \lor (-a_1^w_2 < 0 \land -a_2^w_2 < 0)
\end{align*}
\]  
(36)

Please note that any attempt to solve the system of inequalities fails because the first inequalities of (36) and the latter two inequalities of (34) are incompatible. Then, in this case, two hyper-planes belonging to the convex hull of a multi-box region are always viable and the multi-box inner approximation succeeds in making the resulting approximated region viable.

V. CONCLUSIONS

In this paper, a distributed CG scheme has been proposed for the supervision of dynamically coupled interconnected linear systems subject to local and global constraints and used for solving constrained coordination problems in networked control systems.

A sequential distributed strategy has been proposed and its stability, feasibility and liveness properties analyzed in full details. These results are encouraging and stimulate further researches on the topic, especially towards reducing conservativeness and extending the proposed methodology to more general and complex large-scale examples, eventually in the presence of communication time-delays and packet losses.

Novel insights on the liveness analysis of the scheme have been provided and still open issues requiring further investigations discussed. In particular, the approximation procedure presented in [2] able to achieve arbitrarily accurate viable inner approximations of the original constraint set for mono-dimensional problems has been adapted to the S-CG algorithm and it has been shown, via counter-examples, that its direct application to multi-dimensional problems does not lead to viable sets. In particular this issue remains on open problem for future research works.

REFERENCES